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A NOTE ON THE STRUCTURE OF GRADED MODULES OVER A POLYNOMIAL RING

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Introduction

In the first section of this paper the complete Hilbert functions for graded modules over a polynomial ring $K[X_1, \ldots, X_m]$ where K is a field are studied. The principal results are Theorems 1.1 and 1.2. In the second section, 1.1 is applied to graded modules for the more general case where K is an arbitrary commutative ring. Theorem 1.2 resembles the main result of [1] and is included since it follows easily from the theory developed to prove 1.1. The main accomplishment of the second section is the development of a criterion for determining when a finitely generated graded module over a polynomial ring that happens to be flat over the coefficient ring is finitely presented.

Notation. If A is an integral domain, qf A denotes the quotient field of A. We let $k(p) = \text{qf } R/p$ whenever p is a prime ideal of the ring R. For the benefit of those who do not like to read sequentially an index of the definitions used appears at the end.

1. Dimension functions

A *dimension function* is any function $f : \mathbf{Z} \to \mathbf{N}$ with the following properties:

(1) There exists $a \in \mathbf{Z}$ such that $f(n) = 0$ if $n < a$;

(2) There exists $b \in \mathbf{Z}$ and a polynomial $g(r)$ with coefficients in $\mathbf{Q}$ such that $f(r) = g(r)$ whenever $r > b$.

We define the *degree* of the dimension function f to be the degree of the polynomial g (as in 2 of the above definition). If degree $g \leq m$ and we write the coefficient of r^m in g as $n/m!$, a simple induction shows that $n \in \mathbf{N}$ (for indeed the coefficient of r^{m-1} in $g(r) - g(r-1)$ is $n/(m-1)!$). We call n the *type* of f when it is understood in advance that f is of degree $\leq m$. If f_1 and f_2 are dimension functions, we shall write $f_1 \leq f_2$ if $f_1(r) \leq f_2(r)$ for all r in $\mathbf{Z}$.

If K is any ring we shall let S_K be the polynomial ring $K[X_1, \ldots, X_m]$ where $X_1, X_2, \ldots, X_m$ is a set of indeterminates that we shall hold fixed from now on. We shall let T_X denote the set of all monomials in $X_1, \ldots, X_m$. For $h \in \mathbf{Z}$ let Gr_h be the class of all graded S_K-modules which are generated by finitely many elements homogeneous of degree $\leq h$ where it is assumed that K varies over the class of all fields. It is well known that if M is a finitely generated graded S_K-module and K a field, the function $\dim M$ defined by $(\dim M)(n) = \dim_K M_n$, $n \in \mathbf{Z}$, is a dimension function, and its degree is $<m$. (This in fact follows immediately from 1.1.3 and 1.1.5 though that proof is fundamentally different from the usual one.) Given $h \in \mathbf{Z}$ let F_h denote the set of all dimension functions $\dim M$ for M in Gr_h.

1.1. Theorem. *Let $h \in \mathbf{Z}$ and let $f \in F_h$. Then there exists a number $r \in \mathbf{Z}$ such that if $g \in F_h$ and $g(s) \leq f(s)$ whenever $s \leq r$, then $g \leq f$.*

By hypothesis $f = \dim M$ where for some field K, M is a graded S_K-module generated by finitely many elements homogeneous of degree $\leq h$. If we fix M and use it to define r $(=r(M))$, the value of r obtained will certainly be $\geq h$. Therefore to obtain an r that depends only on f (and not on how M is chosen) we can let $r = \min\{r(M) : \dim M = f\}$. The number r that is given by 1.1 will be denoted by $r_0(f, h, m-1)$ ($m-1$ indicates the degree of f). The function f_h defined by $f_h(s) = f(h+s)$ is $\dim M(h)$ where $M(h)$ is the graded S_K-module isomorphic to M as an S_K-module but graded by the rule $(M(h))_n = M_{h+n}$. We note that $M(h)$ is generated by its elements that are homogeneous of degree ≤ 0 and so $f_h \in F_0$. Suppose we find r in $\mathbf{Z}$ such that when $g' \in F_0, f_h(s) \geq g'(s)$ for $s \leq r-h$ implies $f_h \geq g'$. Then it will follow that $g \in F_h$ and $f(s) \geq g(s)$ for all $s \leq r$ implies $f_h(s) \geq g_h(s)$ for all $s \leq r-h$ and that $g_h \in F_0$, thus $f_h \geq g_h$, and so $f \geq g$. Therefore we only need to prove 1.1 for the case $h=0$. Furthermore $M' = \sum_{n \geq 0} M_n$ is a graded S_K-module generated by finitely many elements homogeneous of degree zero. If for g in F_0 we define $g'(s) = g(s)$ for $s \geq 0$, $g'(s) = 0$ for $s < 0$ we have that $f' = \dim M'$. Evidently if we have r such that $u \in F_0$ and $f'(s) \geq u(s)$ for $s \leq r \Rightarrow f' \geq u$, then $f(s) \geq g(s)$ for $s \leq r$ and $g \in F_0 \Rightarrow f'(s) \geq g'(s)$ for $s \leq r$, so $f' \geq g'$ and so $f \geq g$ (as $r \geq 0$). We may therefore assume that $f(s) = 0$ if $s < 0$ and that therefore M is generated by M_0.

Consider an S_K-module F that is free on elements $e_1, \ldots, e_n$ homogeneous of degree zero. We let $B = \{X^u e_j : u \in \mathbf{N}^m, 1 \leq j \leq n\}$ where $X^u = X_1^{u_1} \cdots X_m^{u_m}$ and note that there are natural bijections $\mathbf{N}^m \times \{1, \ldots, n\} \cong B \cong \coprod_n \mathbf{N}^m$ (=the disjoint union of n copies of $\mathbf{N}^m$). We write $(u, j) \leq (v, k)$ if $u, v \in \mathbf{N}^m$, $j = k$ and $u_i \leq v_i$ for $1 \leq i \leq m$.

Let $|u|_i = u_1 + \cdots + u_i$ if $u \in \mathbf{N}^m$ and $1 \leq i \leq m$. We shall write $|u|_m$ as $|u|$ and call this number the *degree* of u. We write $(u, j) \prec (v, k)$ if $(|u|_m, |u|_{m-1}, \ldots, |u|_1, j)$ precedes $(|v|_m, |v|_{m-1}, \ldots, |v|_1, k)$ in the lexiocographic order on $\mathbf{N}^m \times \{1, \ldots, n\}$ and $(u, j) \neq (v, k)$. We note $u < v \Rightarrow u \prec v$. The two orders that were just defined on $\mathbf{N}^m \times \{1, \ldots, m\}$ induce analogous orders and a notion of degree on B and $\coprod_n \mathbf{N}^m$ since all these sets are isomorphic and $<$ and $\prec$ will be used to denote them also.

If $0 \neq H \in F$ we can write $H = cb + R$ where $0 \neq c \in K, b \in B$ and $R \in \sum_{b' \in B, b' \prec b} Kb'$, and the pair (b,c) is unique. We cal b the *leader* of H and c the *leading coefficient* of H.

1.1.1. Lemma. *Let $U\subset \mathbf{N}^m\times\{1,\dots,n\}$ be infinite. Then there exists an infinite sequence of elements $u_1<u_2<\cdots$ of elements of U.*

Obviously it is enough to prove this in the case $n=1$, e.g. when $U\subset\mathbf{N}^m$. That case is proved by a routine induction (cf. [2]).

If $U\subset B$ an element u of U will be called *primordial* if there is no u' in U with $u'<u$. Let U_- denote the set of primordial elements of U. By the above lemma, $\# U_-$ is finite. If $u\in U$ let u' be the first element for the order $\prec$ of the set $\{u'\in U: u'\leq u\}$. Then $u'\in U_-$, so every element of U is $\geq$ some element of U_-. We call U *spreading* if $u\geq u'\in U \Rightarrow u\in U$. Then U is spreading if and only if $U=\{b\in B: b\geq u$ for some $u\in U_-\}$. We shall assume definitions analogous to this one are made for subsets of $\mathbf{N}^m\times\{1,\dots,n\}$ and $\coprod_n \mathbf{N}^m$. For brevity we shall refer to subsets of $\mathbf{N}^m\times\{1,\dots,n\}$ or B or of a finite disjoint union of copies of $\mathbf{N}^m$ as *m-dimensional* sets.

Consider an exact sequence $0\to N\to F\to M\to 0$ of graded S_K-graded modules where F is free on elements $e_1,\dots,e_n$ homogeneous of degree zero. Let L be the set of all leaders of non-zero elements of N. Since $u\prec v$ implies $u\prec X_iu\prec X_iv$ if $u,v\in B$, it is clear that L is a spreading m-dimensional set. Let $M'=\sum_{b\in B\setminus L}Kb$.

1.1.2. Lemma. $F=M'\oplus N$.

Evidently $M'\cap N=0$ since every non-zero element of N has its leader in L and so can't be in M'. To show that $F=M'+N$ it will suffice to show that $B\subset M'+N$. If $B\not\subset M'+N$, let b be the first element of B with respect to the order $\prec$ not in $M'+N$. Since $B\setminus L\subset M'+N$, $b\in L$. Therefore b is the leader of an element H of N and we may assume that $H=b+R$ where R is a linear combination over K of elements b' of B with $b'\prec b$. By the hypothesis on b each such $b'\in M'+N$, so $b=H-R\in N+M'+N=M'+N$, a contradiction.

If V is an m-dimensional set and $r\in\mathbf{N}$, let $V(r)$ be $\{v\in V: |v|=r\}$ whenever $r\in\mathbf{Z}$. Let $(\# V)(r)=\#(V(r)), r\in\mathbf{Z}$. We have $\# B(r)=n(\mathrm{bin}(m-1+r,m-1))$ where $\mathrm{bin}(p,q)=p(p-1)\cdots(p-q+1)/q!$ when $p\geq q>0$, $\mathrm{bin}(p,0)=1$ if $p\geq 0$, and $\mathrm{bin}(p,q)=0$ if $p<q$ or if $q<0$. By 1.1.2

$$\dim M_r=\dim M'_r=\#(B\setminus L)(r)=n[\mathrm{bin}(m-1+r,m-1)]-\# L(r).$$

The following is now immediate.

1.1.3. Lemma. *If $f\in F_0$ and $f(0)\leq n$, there exists a spreading subset V of $\coprod_n\mathbf{N}^m$ such that $f(r)=n[\mathrm{bin}(m-1+r,m-1)]-\# V(r)$ whenever $r\geq 0$.*

From 1.1.3 it is easy to see that our theorem will result from the following theorem.

1.1′. Theorem. *Let V be a spreading m-dimensional set. There exists r in $\mathbf{Z}$ such that if W is also a spreading m-dimensional set and $\#W(s) \geq \#V(s)$ for $s \leq r$, then $\#W \geq \#V$.*

When V is contained in a finite disjoint union of copies of $\mathbf{N}^m$, say $V = V_1 \amalg \cdots \amalg V_n$ where each V_j is contained in a copy of $\mathbf{N}^m$, we shall call each V_j a *summand* of V. We note that 1.1′ is obvious for the case $m=1$ as we can let r be the maximum of the $\{|v| : v \in V_-\}$.

If $v \in \mathbf{N}^m$ we let $F(v)$, the *fan* of v, be $\{w \in \mathbf{N}^m : w \geq v\}$. When v belongs to a disjoint union of copies of $\mathbf{N}^m$ it is to be understood that $F(v)$ is entirely contained in the summand that contains v. We note that $\#(F(v))(r) = \mathrm{bin}(m-1+r-|v|, m-1)$. If V is an m-dimensional spreading set, any subset of V that can be mapped in a one-to-one degree-preserving manner onto an $(m-1)$-dimensional spreading set will be called a *cut* of V. If V is an m-dimensional spreading set with exactly n distinct non-empty summands, we shall say that V is of *type n*.

1.1.4. Lemma. (1) *Let V be a spreading m-dimensional set whose summands are $V_1, \ldots, V_n$ and let a be in $\mathbf{N}^n$. Then $V_a = \bigcup_{j=1}^n \{v \in V_j : v_m = a\}$ is a cut of V.*

(2) *Let V be a spreading set of type n. Then if $v_1, \ldots, v_n$ lie in distinct summands of V, $V \setminus \bigcup_{j=1}^n F(v_j)$ is a cut of V.*

For proving (1) or (2) of 1.1.4 we may assume $V \subset \mathbf{N}^m$. For (1), observe that the map $V_a \to \mathbf{N}^{m-1}$ defined by $v \mapsto (v_1 + a, v_2, \ldots, v_{m-1})$ does what is required. For (2) we need to show that $v \in V \Rightarrow V \setminus F(v)$ is a cut of V. Whenever $1 \leq i \leq m$ and $0 \leq c < v_i$ define $V_{ic} = \{w \in V : w_i = c \text{ and } w_j \geq v_j \text{ if } i < j \leq m\}$. This defines exactly $v_1 + \cdots + v_m$ different sets V_{ic}. If $w \in V_{ic} \cap V_{jd}$ we cannot have $j > i$ as then $d = w_j \geq v_j$ whereas we are assuming $d < v_j$. Since by symmetry we cannot have $j < i$ either it follows that $j = i$ and so $c = w_i = w_j = d$. Thus $V_{ic} \cap V_{jk} \neq \emptyset \Rightarrow (i,c) = (j,d)$. If $w \in \mathbf{N}^m \setminus F(v)$, $w_i < v_i$ for some i and if i is taken as large as possible, $w \in V_{ic}$ where $c = w_i$. It follows that $V \setminus F(v) = \bigcup V_{ic}$. If $i > 1$,

$$v \mapsto (v_1 + c, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m)$$

defines a degree-preserving map of V_{ic} onto a spreading subset of $\mathbf{N}^{m-1}$ and

$$v \mapsto (v_2 + c, v_3, \ldots, v_m)$$

does the same for V_{1c}. This proves (2) of 1.1.4.

1.1.5. Lemma. *If $V \neq \emptyset$, $\#V$ is a dimension function of degree $m-1$ and the type of $\#V$ is the type of V.*

To establish 1.1.5 for any particular value of m it is enough to show that when $\emptyset \neq V \subset \mathbf{N}^m$, $\# V$ is a dimension function of degree $m-1$ and type 1. For $m=1$, $\# V(r) = \text{bin}(r-s, 0)$ where s is the first element of V. For $m>1$ we can assume 1.1.5 is established for all smaller values of m. Let $v \in V$. Then $V \backslash F(v)$ is a cut W by 1.1.4. Then $\# V(r) = (\# F(v))(r) + \# W(r)$ and by our inductive assumption $\# W = 0$ or is a dimension function of degree $m-2$. Thus

$$\# V(r) = \text{bin}(m-1+r-|v|, m-1) + \# W(r).$$

Since $\text{bin}(m-1+r-|v|, m-1)$ is of degree $m-1$ and has type 1, the same is true for $\# V(r)$ and so 1.1.5 follows.

1.1.6. Remark. A closer look at this last proof shows we can find a finite ordered set S and a function

$$(f_1, f_2) : S \to \{1, \dots, m\} \times \mathbf{N}$$

with the following properties:

(1) $\# V(r) = \sum_{s \in S} \text{bin}(m - f_1(s) + r - f_2(s), m - f_1(s))$.

(2) $s < s'$ implies $f_1(s) \leq f_1(s')$ and $f_2(s) \leq f_2(s')$.

To prove 1.1′ for $m>1$ set $n = \text{type}\, \# V$ and note that $\# \mathbf{N}^m(r) = \text{bin}(m-1+r, m-1)$ has type 1, so for some a in $\mathbf{N}$ we have $(n-1)\# \mathbf{N}^m(a) < \# V(a)$. Thus if W is any m-dimensional spreading set and $\# W(a) \geq \# V(a)$, $n-1$ summands of W cannot contain $W(a)$. Put another way, there will exist points $w_1, \dots, w_n$ of $W(a)$ that lie in distinct summands of W. In particular for our fixed V we can fix elements $v_1, \dots, v_n$ of $V(a)$ that lie in distinct summands $V_1, \dots, V_n$ of V respectively. The other summands of V are then empty so we can assume $V = V_1 \amalg \cdots \amalg V_n$. Then by 1.1.4, $V' = V \backslash \bigcup_{j=1}^n F(v_j)$ is a cut of V. Thus $\# V'$ is the dimension of an $(m-1)$-dimensional spreading set. We have also that

$$V = F(v_1) \amalg F(v_2) \amalg \cdots \amalg F(v_n) \amalg V'.$$

We may assume that the theorem holds for all smaller values of m and thus that there exists b in $\mathbf{N}$ such that if W' is any finite disjoint union of cuts of $\mathbf{N}^m$, $\# W'(s) \geq \# V'(s)$ for $s \leq b$ implies $\# W' \geq \# V'$. Let $r = \sup(a, b)$.

Assume now that W is as in the statement of 1.1′ with r as we have chosen it. We need to show that $\# W \geq \# V$. Since $\# W(a) \geq \# V(a)$, it follows (as was noted above) that there exist $w_1, \dots, w_n$ in $W(a)$ lying in distinct summands $W_1, \dots, W_n$ of W respectively. Let $W_{n+1}, \dots, W_h$ be the remaining summands of W. Let

$$W' = \bigcup_{j=1}^{n} [W_j \backslash F(w_j)]$$

and let

$$W'' = \{z \in W_{n+1} \cup \cdots \cup W_h : z_m \leq r\}.$$

Then

$$W \supset Z = F(w_1) \cup \cdots \cup F(w_n) \cup W' \cup W''$$

and this union is disjoint. It will suffice to show that $\#Z \geq \#V$. Now if $s \leq r$, $Z(s) = W(s)$, so $\#(W' \cup W'')(s) \geq \#V'(s)$ and by 1.1.4 $W' \cup W''$ and V' are cuts. Also $b \leq r$. Therefore, by the reasoning noted above, our inductive hypothesis implies $\#(W' \cup W'') \geq \#V'$ so

$$\#Z = \sum_{j=1}^{n} \#F(w_j) + \#(W' \cup W'') \geq \sum_{j=1}^{n} \#F(v_j) + \#V' = \#V.$$

Unlike the theorem in [1], m is fixed in the following result.

1.2. Theorem. *If $h \in \mathbf{Z}$, F_h contains no infinite strictly decreasing sequence.*

If $M^1, M^2, \ldots$ is an infinite sequence of members of Gr_h such that $f_1 > f_2 > \cdots$ where $f_j = \dim M^j$ then the sequence $f_1' \geq f_2' \geq \cdots$ where $f_j'(r) = f_j(r+h)$ if $r \geq 0$, $f_j'(r) = 0$ if $r < 0$ has an infinite strictly decreasing subsequence. Also $f_j' = \dim M'^j$ where $M'^j = \sum_{r \geq 0} M_{r+h}^j$. Therefore in looking for a contradiction we can assume that $h = 0$ and also that $f_j(r) = 0$ for every j if $r < 0$. Also evidently (by omitting a finite number of the f_j) we may assume that all $f_j(0)$ have the same value n. Then $\dim M^j = n \# \mathbf{N}^m - \# V_j$ by 1.1.3 where V_j is a spreading subset of $\coprod_n \mathbf{N}^m$. It will suffice to show that $\# V_j$ cannot increase indefinitely. The number of non-empty summands of V_j eventually is constant and we shall now redefine n to be that constant number. We shall also discard the (finitely many) V_j with fewer than n summands. Then we have $a \in \mathbf{N}$ such that $\# V_j(a) > (n-1)[\mathrm{bin}(m-1+a, m-1)]$ for all j. Considering now any particular value of j we note there exist elements $v_1, \ldots, v_n$ of $V_j(a)$ that lie in distinct summands. Therefore

$$V_j = V_j' \cup F(v_1) \cup \cdots \cup F(v_n)$$

where $V_j' = V_j \setminus \bigcup_{h=1}^n F(v_h)$ and evidently this union is disjoint. Examination of the proof of 1.1.4 shows that V_j' has a degree-preserving isomorphism with a spreading subset of $\coprod_{na} \mathbf{N}^{m-1}$. After making these definitions for each j we shall have $\# V_1' < \# V_2' < \cdots$ where the V_j' are essentially spreading subsets of $\coprod_{na} \mathbf{N}^{m-1}$. That contradicts the case $m-1$ that we can assume (by induction) already to be established.

2. Applications to graded modules over a general polynomial ring

If R is any (commutative) ring an *R-field* is any homomorphism of rings $\phi : R \to K$ with K a field. We usually take ϕ for granted and refer to K as the R-field. The following lemma is recalled for the reader's convenience.

2.1. Lemma. *Let S be any ring (commutative or not) and M an S-module. The following are equivalent:*

(1) *There is a surjection $F \to M$ of left S-modules such that F is finitely generated free and the kernel is a finitely generated submodule of F;*

(2) *M is finitely generated and any surjection $F \to M$ of S-modules where F is finitely generated has a finitely generated kernel.*

A module which has the equivalent properties of 2.1 is called *finitely presented.* A reference for 2.1 is [3].

An R-module is called *regular* if it is finitely generated and projective. A finitely generated graded S_R-module M is called *regular* if every M_r is a regular R-module (which does not imply it is regular when considered as a non-graded S_R-module). If M is any finitely generated graded S_R-module and K is an R-field, let $f_{M,K}(r) = \dim_K(K \otimes_R M_r)$ for every r in $\mathbf{Z}$. We call $f_{M,K}$ a *dimension function* of M, and note that it is indeed a dimension function in the sense of the definition given previously. If $p \in \operatorname{Spec} R$ we let $f_{M,p} = f_{M,K}$ where $K = \operatorname{qf} R/p$.

2.2. Theorem. *Let M be a finitely generated graded S_R-module and consider the following properties of M:*

(1) *M is finitely presented as an S_R-module;*

(2) *M has only finitely many dimension functions.*

Then (1)$\Rightarrow$(2) *and, if M is regular,* (2)$\Rightarrow$(1).

2.2.1. Lemma. (1) *Let M be a finitely generated module over a ring R, p a prime ideal of R and $K = \operatorname{qf} R/p$. If $\dim_K K \otimes_R M = n$, there exists a g in $R \setminus p$ such that $M[1/g] = R[1/g] \otimes_R M$ is generated as a $R[1/g]$-module by n elements.*

(2) *If furthermore M is regular, g can be chosen so that $M[1/g]$ is a free $R[1/g]$-module on n generators.*

To prove 2.2.1 let F be a free R-module on n generators $e_1, \ldots, e_n$ and let $x_1, \ldots, x_n$ in M be chosen so that their images in $k(p) \otimes_R M$ generate that $k(p)$-vector space. Let $\phi : F \to M$ be the R-module map that sends e_j to x_j for each j. Since

$$k(p) \otimes \phi : k(p) \otimes_R F \to k(p) \otimes_R M$$

is an isomorphism, the cokernel C of $\phi_p : F_p \to M_p$ is zero and so $C[1/g]$ is zero for some g in $R \setminus p$. That makes $\phi[1/g]$ surjective and (1) of 2.2.1 therefore follows. To prove (2) note that the exact sequence

$$0 \to N \to F \to M \to 0$$

where $N = \ker \phi$ implies an exact sequence

$$0 \to k(p) \otimes_R N \to k(p) \otimes_R F \to k(p) \otimes_R M \to 0$$

so $k(p) \otimes_R N = 0$ which implies that $N[1/g] = 0$ for some g in $R \setminus p$, and (2) follows.

In proving 2.2 the implication (1)$\Rightarrow$(2) will be shown first. If M is a finitely presented graded S_R-module it is well known (and easy to prove) that $M \cong R \otimes_{R'} M'$ where R' is a finitely generated algebra over $\mathbf{Z}$ and M' is a finitely generated $S_{R'}$-module. Evidently any dimension function of M is also a dimension function of M', so to show that M has only finitely many dimension functions it will suffice to show the same is true for the R'-module M'. Thus we can assume to begin with that R is noetherian.

Let p_1 be any minimal prime ideal of R and $p_2, \dots, p_h$ the others. Assume that M is generated by its elements that are homogeneous of degree $\le h$ and let r be the number of 1.1 for h and the dimension function $f = \dim k(p_1) \otimes_R M$. As $f(s) \neq 0$ for only finitely many $s \le r$ we may use 2.1.1 to choose an element g of $R \setminus p_1$ such that $M_s[1/g]$ is generated as an $R[1/g]$-module by $f(s)$ elements for each $s \le r$. Then if $q \in \operatorname{Spec} R$ and $g \notin q \supset p_1$, $f_{M,q} \ge f$ but also $f_{M,q}(s) \le f(s)$ whenever $s \le r$, so by 1.1, $f_{M,q} = f$. Let I_0 be the ideal $g p_2 \cdots p_h$ of R. If $q \in \operatorname{Spec} R$ and $f_{M,q} \neq f$, $q \supset I_0$. Therefore the dimension functions of M other than f are all dimension functions of the graded S_{R/I_0}-module M/I_0M. Also $I_0 \neq (0)$ because $I_0 \not\subset p_1$.

It is now clear that if R is a noetherian ring and M any finitely generated graded S_R-module with infinitely many dimension functions, there is a (proper) non-zero ideal I_0 of R such that the graded S_{R/I_0}-module M/I_0M has infinitely many dimension functions. It follows, by considering R/I_0 and M/I_0M, that there exists an ideal $I_1 \supsetneqq I_0$ of R with $I_1 \neq R$ such that M/I_1M is a graded S_{R/I_1}-module with infinitely many dimension functions. Indefinite repetition of this procedure gives an infinite strictly increasing sequence $I_0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \cdots$ of ideals of R, contradicting the fact that R is noetherian. It follows that M cannot have infinitely many dimension functions.

Before we show the implication (2) $\Rightarrow$ (1) of 2.2 (for M regular) let us note that the regularity assumption is needed. For this we can let R be a ring with only one prime ideal where that ideal is not finitely generated but does have a sequence of generators $a_0, a_1, \dots$. Let $m = 1$, $X = X_1$ and let $M_r = R/(a_0, \dots, a_r)$, $r \in \mathbf{N}$. Let $X : M_r \to M_{r+1}$ be induced by id_R for all r and let $M = \sum M_r$. Then M has only one dimension function but evidently is not finitely presented.

Assume now that M is regular, is generated by $\bigcup_{s \le h} M_h$, and has finitely many dimension functions $f_1, \dots, f_q$. By 1.1 there exists r in $\mathbf{N}$ such that if $f \in F_h$, $f \ge f_i$ for some i with $f(s) = f_i(s)$ for all $s \le r$ then $f = f_i$. Let $u : P \to M$ be a surjection of graded S_R-modules where P is finitely generated and free on elements of degree $\le h$. Let $N = \ker u$. Let $N' \subset N$ be the S_R-module of N generated by all the N_s with $s \le r$. Then N' is finitely generated. Let $M' = P/N'$ and let $\upsilon : M' \to M$ be the canonical map. It will suffice to show that υ is an isomorphism. If K is any R-field, $\mathrm{id}_K \otimes_R \upsilon$ is a surjective homomorphism of graded S_K-modules which have identical dimension functions (because of the way r was picked) so it is an isomorphism. Thus for n in $\mathbf{Z}$, $\mathrm{id}_K \otimes \upsilon_n$ (where $\upsilon_n : M_n' \to M_n$) is an isomorphism. It follows that if $p = \operatorname{Ann}_R K$, $\mathrm{id}_{R_p} \otimes_R \upsilon_n$ is an isomorphism since by the flatness assumption $(M_n)_p$ is a free module over R_p. Now as p is an arbitrary prime ideal of R, υ_n is an isomorphism. Thus υ is an isomorphism, so $N' = N$ and M is finitely presented.

Because of (2) of 2.1.1 it is clear that if M is a finitely presented flat R-module and C is a connected component of Spec R, $\dim_{k(p)} k(p) \otimes_R M$ is constant for p in C. It follows that if M is a regular graded S_R-module the $f_{M,p}$ for p in C are all the same. The following is therefore an immediate corollary of 2.2.

2.3. Corollary. *If* Spec R *has only finitely many connected components, then many finitely generated regular graded* S_R*-module is finitely presented.*

The following example shows that the hypothesis of 2.3 is needed.

2.4. Example. Let $k_0, k_1, \ldots$ be an infinite sequence of fields all isomorphic to a given field k, and let R be $\prod_{r\geq 0} k_r$ considered as a ring in the usual way. For r in $\mathbf{N}$ let e_r be the element of R defined by $(e_r)_s = 0$ for $s<r$ and $=1_{k_s}$ for $s \geq r$. Let M_r be the ideal of R generated by e_r and define $X: M_r \to M_{r+1}$ by $Xe_r = e_{r+1}$ for all r in $\mathbf{N}$. Then $M = \sum_{r\geq 0} M_r$ is a graded $R[X]$-module and is generated by the single element $(e_0, 0, 0, \ldots)$ of M_0. If $r \in \mathbf{N}$, $p_r = \{a \in R: a_r = 0\}$ is a maximal ideal of R and $k(p_r) \cong k_r$. Also $k_r \otimes_R M_s / p_r M_s \cong k$ or 0 accordingly as $s \leq r$ or $s > r$. The dimension function $f_r = f_{p_r}$ is therefore given by $f_r(s) = 1$ or 0 accordingly as $s \leq r$ or $s > r$. Thus M is not finitely presented. However each M_r is a regular R-module because $R = N_r \oplus M_r$ where N_r is the ideal of R consisting of all those elements R with $a_s = 0$ for all $s \geq r$.

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Except as noted each entry of this index is referenced by the first result that follows it in the text.

References

[1] W. Sit, Well-ordering of certain numerical polynomials, Trans. Amer. Math. Soc. 212 (1975) 37–45.

[2] E. Kolchin, Differential algebra and algebraic groups, Pure and Applied Mathematics, Vol. 54 (Academic Press, New York, 1973) p. 49, Lemma 15 (a).

[3] J. Rotman, Notes on Homological Algebra (Van Nostrand–Reinhold, New York, 1970) p. 62. (This reference uses 'finitely related' in place of 'finitely presented'.)